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BUCKLING OF CIRCULAR CYLINDRICAL SHELLS WITH EVENLY SPACED, EQUAL STRENGTH CIRCULAR RING FRAMES

PART II

by

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LIST OF SYMBOLS

a	radius of middle surface of shell
C_r	St. Venant torsion constant
$D =$	$Eh^3/12(1 - \nu^2)$
e	eccentricity of centroid of ring frame section
E	modulus of elasticity
G	shear modulus
h	thickness of shell
$I_{x_o}, I_{z_o}, I_{x_oz_o}$	moments and product of inertia of ring section
$k =$	$h^2/12a^2$
$K =$	$Eh/(1 - \nu^2)$
L	distance between ring frames
$L_o =$	L/a
m	number of waves in circumferential direction
n	number of half waves in axial direction
$M =$	$2I_{z_o}(1 - \nu^2)/Lha^2$
$N =$	$C_r(1 - \nu)/Lha^2$
p	radial pressure in lbs. per sq. in.
P	axial pressure in lbs. per sq. in.
$Q =$	$2\Gamma(1 - \nu^2)/Lha^4$
$R_A =$	$m^2(m^2M + N + m^2Q)$
$R_{AC} =$	$m^2\lambda[M(1 - m^2S) + (N + m^2Q)(1 - S)]$
$R_C =$	$\lambda^2[M(1 - m^2S)^2 + m^2(N + m^2Q)(1 - S)^2]$
$S =$	(e/a)

u, v, w		displacement components of median surface of shell
U_e		extensional strain energy of shell
U_b		bending strain energy of shell
U_r		strain energy of ring frame
V_1		potential due to axial pressure P
V_2		potential due to radial pressure p
x, y, z		coordinates of shell and ring
Γ		warping constant
λ	=	$(n\pi a)/L$
ν		Poisson's ratio
ξ	=	x/a
φ		angular coordinate in circumferential direction
Φ_1	=	(pa/K)
Φ_2	=	(Ph/K)

SUMMARY

A strain-energy solution of the buckling of a circular cylindrical shell reinforced by evenly spaced circular ring frames of equal strength under hydrostatic pressure was obtained in Ref. (1). The buckled shape was assumed to be sinusoidal with inflection points at the location of the ring frames. For several geometrical configurations the critical pressure was found to be from two to three times that given by the von Mises solution (Ref. 2).

The assumed buckled shape used in Ref. (1) was modified to permit the inflection points of the deformed shape to occur between ring frames. For the geometrical configurations calculated in Ref. (1), the critical pressure was found to be about one and one-third times that given by the von Mises solution. This represents a marked improvement over the solution obtained in Ref. (1).

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RESUME OF STRAIN ENERGY AND POTENTIAL OF CYLINDRICAL SHELL

The extensional and bending strain energies of the shell are given by equation (3), (4) and (29) of Ref. (1) as

$$U_e = [Eh/2(1-\nu^2)]\int_0^{2\pi}\int_0^{L_0}[u_\xi^2+(v_\varphi-w)^2+2\nu u_\xi(v_\varphi-w)+\{(1-\nu)/2\}(v_\xi+u_\varphi)^2]d\xi d\varphi \tag{1}$$

$$U_b = [Ehk/2(1-\nu^2)]\int_0^{2\pi}\int_0^{L_0}[w_{\xi\xi}^2 + (w_{\varphi\varphi}+w)^2 + 2\nu w_{\xi\xi}(w_{\varphi\varphi}+w) + 2(1-\nu)\{w_{\xi\varphi}+(1/2)v_\xi-(1/2)u_\varphi\}^2]d\xi d\varphi \tag{2}$$

The strain energy stored in the ring frames as given in Ref. (1) equation (13) is

$$U_r = \Big\{[EI_{x_0}/2a^3]\int_0^{2\pi}(w+w_{\varphi\varphi})^2 d\varphi+[EI_{z_0}/2a^3]\int_0^{2\pi}[w_\xi-u_{\varphi\varphi}+(e/a)w_{\xi\varphi\varphi}]^2 d\varphi + [EA'/2a]\int_0^{2\pi}w^2 d\varphi+[C_rG/2a^3]\int_0^{2\pi}[w_{\xi\varphi}(1-e/a)+u_\varphi]^2 d\varphi + [E\Gamma/2a^5]\int_0^{2\pi}[w_{\xi\varphi\varphi}(1-e/a)+u_{\varphi\varphi}]^2 d\varphi\Big\}_{\xi=L/a} \tag{3}$$

The potential due to the axial pressure P(lbs. per sq. in.) given in Ref. (1) equation (20) is

$$V_1 = -[Ph/2]\int_0^{2\pi}\int_0^{L_0}(u_\xi^2+v_\xi^2+w_\xi^2)d\varphi d\xi \tag{4}$$

The potential due to the radial pressure p (lbs. per sq. in.) as given in Ref. (1) equation (26) is

$$V_2 = -[pa/2]\int_0^{2\pi}\int_0^{L_0}w(u_\xi-w-w_{\varphi\varphi})d\varphi d\xi \tag{5}$$

ASSUMED BUCKLED SHAPE

The assumed shape for the shell and ring frames used in the previous report [Ref. (1), equation 27] satisfied the boundary condition of simple s pports at each ring frame. For the case of ring frames which have large resistance to twisting and bending this condition is not the most suitable. In order to describe end conditions which are intermediate between simple support and rigid fixation at the ring frame, a term of the form

$$D \cos m\varphi (1 - \cos 2\lambda\xi) \tag{6}$$

must be added to the radial displacement w.

The displacement pattern consequently becomes

$$\begin{aligned} u &= A \cos m\varphi \cos \lambda\xi \\ v &= B \sin m\varphi \sin \lambda\xi \\ w &= C \cos m\varphi \sin\lambda\xi + D \cos m\varphi\ (1 - \cos 2\lambda\xi) \end{aligned} \tag{7}$$

The radial displacement w is zero at the ring frames and has at least two symmetrical points of inflection between them. The displacement pattern assumed in equations (7) above contains four degrees of freedon because of the four arbitrary shape constants A, B, C and D and will lead to a four by four determinant for the calculation of the critical pressure.

If the displacements given by equation (7) and their derivatives are inserted in the energy expressions given by equations (1) to (5) and if the indicated integrations are carried out, the energy expressions become:

$$U_e = (KL_o\pi/4)\Big\{\lambda^2A^2 + (mB-C)^2 - 2\nu\lambda A(mB-C) + [(1-\nu)/2](\lambda B-mA)^2 + $$
$$+ 3D^2 - (16/3\lambda L_o)[1-(-1)^n][(mB-C)-\nu\lambda AD]\Big\} \qquad (8)$$

$$U_b = (kKL_o\pi/4)\Big\{C^2[\lambda^4+(m^2-1)^2+2\nu\lambda^2(m^2-1)]+[(1-\nu)/2][mA+\lambda B-2m\lambda C]^2$$
$$+ D^2[16\lambda^4+8m^2\lambda^2(1-\nu)+3(m^2-1)^2+8\nu\lambda^2(m^2-1)]$$
$$+ (16/3\lambda L_o)\Big[1-(-1)^n][-m\lambda D(1-\nu)(mA+\lambda B-2m\lambda C)$$
$$+ CD[\lambda^4 +(m^2-1)^2+2\nu\lambda^2(m^2-1)]\Big]\Big\} \qquad (9)$$

$$V_1 = -(KL_o\pi/4)(Ph\lambda^2/K)\Big\{A^2+B^2+C^2+4D^2+(16/3\lambda L_o)[1-(-1)^n]CD\Big\} \qquad (10)$$

$$V_2 = -(KL_o\pi/4)(pa/K)\Big\{C^2[m^2-1]-\lambda AC+3D^2(m^2-1)\Big\}$$
$$+ (8/3\lambda L_o)[1-(-1)^n][2CD(m^2-1)-\lambda AD]\Big\} \qquad (11)$$

$$U_r = (KL_o\pi/4)\Big\{(2EI_{z_o}/KL_oa^3)[(1-em^2/a)^2\lambda^2C^2+m^4A^2+2m^2\lambda AC(1-em^2/a)]$$
$$+ (2GC_r/KL_oa^3)[(1-e/a)^2m^2\lambda^2C^2+m^2A^2+2m^2\lambda AC(1-e/a)]$$
$$+ (2E\Gamma/KL_oa^5)[(1-e/a)^2m^4\lambda^2C^2+m^4A^2+2m^4\lambda AC(1-e/a)]\Big\} \qquad (12)$$

where

$$K = Eh/(1-\nu^2)\ , \ k = h^2/12a^2 \qquad (12a)$$

The ring energy may be simplified as in Ref. (1) by making the following substitutions:

$$M = 2EI_{z_o}/KL_oa^3 = 2I_{z_o}(1-\nu^2)/(Lha^2)$$
$$N = 2C_rG/KL_oa^3 = C_r(1-\nu)/(Lha^2)$$
$$Q = 2E\Gamma/KL_oa^5 = 2\Gamma(1-\nu^2)/(Lha^4)$$
$$S = (e/a)$$
$$R_A = m^2(m^2M + N + m^2Q)$$
$$R_{AC} = m^2\lambda[M(1-Sm^2)+(N+m^2Q)(1-S)]$$
$$R_C = \lambda^2[M(1-Sm^2)^2+m^2(N+m^2Q)(1-S)^2]$$

The ring energy then becomes

$$U_r = (KL_o\pi/4)[A^2R_A + 2ACR_{AC} + C^2R_C] \tag{14}$$

which corresponds exactly to equation (33) of Ref. (1).

DETERMINATION OF THE CRITICAL LOAD

The total potential is given by

$$U = U_e + U_b + U_r + V_1 + V_2 \tag{15}$$

In order for equilibrium to exist the variation of the total potential with respect to the parameters A, B, C and D must vanish. The algebraic equations resulting from this differentiation are given below. The equations are symmetrical and homogeneous.

$$\begin{aligned}
&A[\lambda^2 + m^2(1-\nu)(1+k)(1/2) + R_A - Ph\lambda^2/K] \\
&+ B[-\lambda m\{(1/2)(1+\nu) - (k/2)(1-\nu)\}] \\
&+ C[\lambda\nu - (1-\nu)\lambda m^2 k + R_{AC} + pa\lambda/2K] \\
&+ D\left[(8/3\lambda L_o)[1-(-1)^n]\{\nu\lambda - (1-\nu)\lambda m^2 k + pa\lambda/2K\}\right] = 0
\end{aligned} \tag{16}$$

$$\begin{aligned}
&A[-\lambda m\{(1/2)(1+\nu) - (k/2)(1-\nu)\}] \\
&+ B[m^2 + (1-\nu)(1+k)(\lambda^2/2) - Ph\lambda^2/K] \\
&+ C[-m\{1 + (1-\nu)\lambda^2 k\}] \\
&+ D\left[(8/3\lambda L_o)[1-(-1)^n]\{-m[1+\lambda^2 k(1-\nu)]\}\right] = 0
\end{aligned} \tag{17}$$

$$\begin{aligned}
&A[\lambda\nu - (1-\nu)\lambda m^2 k + R_{AC} + pa\lambda/2K] \\
&+ B[-m\{1 + \lambda^2 k(1-\nu)\}] \\
&+ C[1 + k\{(m^2+\lambda^2)^2 + 1 - 2m^2 - 2\nu\lambda^2\} + R_C - Ph\lambda^2/K - pa(m^2-1)/K] \\
&+ D\Big[(8/3\lambda L_o)[1-(-1)^n]\{1 + k[(m^2+\lambda^2)^2 + 1 - 2m^2 - 2\nu\lambda^2] \\
&- Ph\lambda^2/K - pa(m^2-1)/K\}\Big] = 0
\end{aligned} \tag{18}$$

$$
\begin{aligned}
&A\left[(8/3\lambda L_o)[1-(-1)^n]\left\{\lambda\nu-(1-\nu)\lambda m^2k+pa\lambda/2K\right\}\right]\\
&+B\left[(8/3\lambda L_o)[1-(-1)^n]\left\{-m[1+\lambda^2k(1-\nu)]\right\}\right]\\
&+C\left[(8/3\lambda L_o)[1-(-1)^n]\left\{1+k[(m^2+\lambda^2)^2+1-2m^2-2\nu\lambda^2]\right.\right.\\
&\left.\left.-Ph\lambda^2/K-pa(m^2-1)/K\right\}\right]\\
&+D[3+8k\left\{2\lambda^4+3(1-m^2)^2/8+m^2\lambda^2-\nu\lambda^2\right\}-4Ph\lambda^2/K-3pa(m^2-1)/K]=0
\end{aligned}
\tag{19}
$$

In order that a non-trivial solution exist for these equations, the determinant formed from the coefficients of the parameters A,B,C and D must vanish. The determinant is:

$$\begin{vmatrix} a_{11} - \Phi_2\lambda^2 & a_{12} & a_{13} + \Phi_1\lambda/2 & t(a_{13} - R_{AC} + \Phi_1\lambda/2) \\ a_{12} & a_{22} - \Phi_2\lambda^2 & a_{23} & ta_{23} \\ a_{13} + \Phi_1\lambda/2 & a_{23} & a_{33} - \Phi_2\lambda^2 - \Phi_1(m^2-1) & t[a_{33} - R_C - \Phi_2\lambda^2 - \Phi_1(m^2-1)] \\ t[a_{13} - R_{AC} + \Phi_1\lambda/2] & ta_{23} & t[a_{33} - R_C - \Phi_2\lambda^2 - \Phi_1(m^2-1)] & a_{44} - 4\Phi_2\lambda^2 - 3(m^2-1)\Phi_1 \end{vmatrix} = 0$$

where

$$a_{11} = \lambda^2 + m^2(1+k)(1-\nu)/2 + R_A$$
$$a_{12} = -\lambda m\{(1/2)(1+\nu) - (k/2)(1-\nu)\}$$
$$a_{13} = \lambda\nu - (1-\nu)\lambda m^2 k + R_{AC}$$
$$a_{23} = -m\{1 + \lambda^2 k(1-\nu)\}$$
$$a_{22} = m^2 + (1-\nu)(1+k)(\lambda^2/2)$$
$$a_{33} = 1 + k[(m^2+\lambda^2)^2 + 1 - 2m^2 - 2\nu\lambda^2] + R_C$$
$$a_{44} = 3 + 8k\{2\lambda^4 + 3(1-m^2)^2/8 + m^2\lambda^2 - \nu\lambda^2\}$$
$$\Phi_1 = pa/K$$
$$\Phi_2 = Ph/K$$
$$t = (8/3\lambda L_o)[1-(-1)^n]$$

SPECIAL SOLUTIONS OF THE DETERMINANT

For the special case of a cylinder which is infinitely long, $\lambda = (n\pi a/L)$ approaches zero for a finite value of n (number of half waves in the axial direction) and R_A, R_{AC} and R_C also approach zero. The determinant (20) reduces to

$$\begin{vmatrix} m^2(1+k)(1-\nu)/2 & 0 & 0 & 0 \\ 0 & m^2 & -m & -tm \\ 0 & -m & 1+k(m^2-1)^2-\Phi_1(m^2-1) & t[1+k(m^2-1)^2-\Phi_1(m^2-1)] \\ 0 & -tm & t[1+k(m^2-1)^2-\Phi_1(m^2-1)] & 3[1+k(m^2-1)^2-\Phi_1(m^2-1)] \end{vmatrix} = 0 \tag{22}$$

The expansion of this determinant yields

$$m^2[3-t^2][1+k(m^2-1)^2-\Phi_1(m^2-1)]^2-m^2[3-t^2][1+k(m^2-1)^2-\Phi_1(m^2-1)]=0 \tag{23}$$

The two solutions obtained from equation (23) are

$$\Phi_{1a} = \frac{1+k(m^2-1)^2}{(m^2-1)} \tag{24a}$$

$$\Phi_{1b} = k(m^2-1) \tag{24b}$$

It is noted that both solutions given above are independent of n. This means that the buckling pressure for an infinitely long cylinder is independent of the number of half-waves in the axial direction. Since the critical pressure obtained from equation (24b) is lower than the one obtained from equation (24a),

the critical pressure for an infinitely long shell after suitable transformation of equation (24b) is given by

$$p_{cr} = (m^2-1)[Eh^3/12a^3(1-\nu^2)] \tag{25}$$

This agrees with the results given in equation (41) of Ref. (1) and on p. 574 of Ref. (2).

BUCKLING LOAD FOR ODD VALUES OF n

If the third row of determinant (20) is multiplied by t and the result subtracted from the fourth row, and if the positions of rows and columns of the resulting determinant are changed, the determinant given below is obtained.

$$
\begin{vmatrix}
\begin{array}{l} a_{44}-4\lambda^2\Phi_2-3(m^2-1)\Phi_1 \\ -t^2[a_{33}-R_C-\Phi_2\lambda^2-\Phi_1(m^2-1)] \end{array} & t(-R_{AC}) & 0 & t(-R_C) \\
t[a_{13}-R_{AC}+\Phi_1\lambda/2] & a_{11}-\lambda^2\Phi_2 & a_{12} & a_{13}+\Phi_1\lambda/2 \\
t[a_{23}] & a_{12} & a_{22}-\Phi_2\lambda^2 & a_{23} \\
t[a_{33}-R_C-\Phi_2\lambda^2-\Phi_1(m^2-1)] & a_{13}+\Phi_1\lambda/2 & a_{23} & a_{33}-\Phi_2\lambda^2-\Phi_1(m^2-1)
\end{vmatrix} = 0 \tag{26}
$$

This determinant may be expanded more readily than determinant (20).

The complete expansion leads to a fourth degree algebraic equation in Φ_1 and Φ_2. The equation is linearized by neglecting terms of the order of Φ_1^2, Φ_2^2, $\Phi_1\Phi_2$ and higher. The solution obtained when the structure is under hydrostatic pressure, that is, when

$$\Phi_1 = \Phi_2/2$$

is

$$\Phi_1\,[1+\Phi_1^{*}H_1-t^2H_2] = \Phi_1^{*}-t^2H_3 \tag{27}$$

where

$$\Phi_1^* = (1/H)[a_{11}a_{22}a_{33} + 2a_{12}a_{13}a_{23} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2] \quad (28)$$

$$\begin{aligned} H = (\lambda^2/2)[a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{13}^2 - a_{12}^2 - a_{23}^2] + \\ + [a_{11}a_{22} - a_{12}^2](m^2-1) + \lambda[a_{22}a_{13} - a_{12}a_{23}] \end{aligned} \quad (29a)$$

$$H_1 = \frac{(3-t^2)(\lambda^2/2+m^2-1) + \lambda^2/2}{[a_{44}-t^2a_{33}+2t^2R_C]} \quad (29b)$$

$$H_2 = \frac{[R_CR_{AC}(\lambda a_{22}-\lambda^2a_{13})+R_C^2(\lambda^2/2)(a_{11}+a_{22})+R_{AC}^2\{a_{22}(m^2-1+\lambda^2/2)+a_{33}\lambda^2/2\}]}{H[a_{44}-t^2a_{33}+2t^2R_C]} \quad (29c)$$

$$H_3 = \frac{[R_C^2(a_{11}a_{22}-a_{12}^2)+R_{AC}^2(a_{33}a_{22}-a_{23}^2)+2R_CR_{AC}(a_{12}a_{23}-a_{22}a_{13})]}{H[a_{44}-t^2a_{33}+2t^2R_C]} \quad (29d)$$

Equation (28) may be transformed to

$$\Phi_1^* = [C_1+kC_2]/[C_3+C_4/2] \quad (30)$$

which corresponds to equation (44) of Ref. (1) and is the solution for the buckling problem when the displacement pattern of Ref. (1) is used. It is noted that for large values of $\lambda = n\pi a/L$ and for $m = 2$ to $m = 15$ it was shown in Ref. (1) that equation (30) can be approximated by

$$\Phi_1^* = [\lambda^4(1-\nu^2)/(m^2+\lambda^2)^2+k(m^2+\lambda^2)^2+R_C]/[m^2-1+\lambda^2/2] \quad (31)$$

BUCKLING LOAD FOR EVEN VALUES OF n

When there are an even number of half-waves n in the axial direction the expression for t given in equation group (21) becomes zero. Determinant (20) reduces to:

$$\begin{vmatrix} a_{11}-\Phi_2\lambda^2 & a_{12} & a_{13}+\Phi_1\lambda/2 & 0 \\ a_{12} & a_{22}-\Phi_2\lambda^2 & a_{23} & 0 \\ a_{13}+\Phi_1\lambda/2 & a_{23} & a_{33}-\Phi_2\lambda^2-\Phi_1(m^2-1) & 0 \\ 0 & 0 & 0 & a_{44}-4\Phi_2\lambda^2-3(m^2-1)\Phi_1 \end{vmatrix} = 0 \qquad (32)$$

In symbolic form the expansion may be written as

$$[a_{44}-4\Phi_2\lambda^2-3(m^2-1)]\,|F| = 0 \qquad (33)$$

where $|F|$ is the determinant given as equation (39) in Ref. (1), which was obtained from a consideration of the simple sinusoidal displacement pattern assumed there. Two solutions are obtained by setting each factor in equation (33) equal to zero. For the hydrostatic case the two solutions are

$$\Phi_1' = \frac{3+k[16\lambda^4+3(m^2-1)^2+8\lambda^2m^2-8\lambda^2\nu]}{3(m^2-1)+(2\lambda^2)} \qquad (34)$$

and

$$\Phi_1^* = [C_1+kC_2]/[C_3+C_4/2] \qquad (35)$$

Equation(35) is the same as equation (30) and may also be approximated by equation (31) for large values of λ and for values of m form 2 to 15.

DETERMINATION OF MINIMUM BUCKLING LOAD AND ASSOCIATED CONFIGURATION

In order to obtain the smallest value of the buckling load it is necessary to minimize equations (27), (32) and (33) with respect to m and n. To do this formally requires extended algebraic manipulations. A more practical method for obtaining the smallest value of the buckling load is given below.

It is known that the buckling load of a cylinder whose ends are partially fixed lies between the value given by von Mises (Ref. 3) for a simply supported shell and that obtained for a shell whose ends are rigidly fixed. Both these solutions are relatively simple in form and may easily be minimized numerically to give good approximate values of m and n for a prescribed shell geometry. These values of m and n are then refined numerically by using equations (27), (32) and (33) which contain the effect of partial end fixity of the shell.

For a shell with simple end supports, R_A, R_{AC} and R_C are zero. In addition, the deformation pattern is sinusoidal with inflection points at the rings so that the constant D in equation (7) is zero. The resulting third order determinant is exactly the same as that obtained previously as equation (39) of Ref. (1). For large values of λ^2 and for the case of hydrostatic pressure, the expansion of the determinant yields

$$[\Phi_1]_{S.S.} = [\lambda^4(1-\nu^2)/(m^2+\lambda^2)^2 + k(m^2+\lambda^2)^2]/(m^2-1+\lambda^2/2) \qquad (36)$$

Equation (36) where the subscript S.S. means simple support corresponds to equation (45) of Ref. (1). A similar expression given in Ref. (3) was obtained by von Mises by using differential equations.

For a shell with rigid supports the deformation pattern is defined by equations (7) with C = 0. The third row and third column of determinant (20) vanish. For the case of hydrostatic pressure the expansion of the reduced determinant yields

$$[\Phi_1]_{R.S.} = (1/F)\left\{a_{44}(\bar{a}_{11}a_{22}-a_{12}^2)+t^2[2a_{12}a_{23}(a_{13}-R_{AC}) - \bar{a}_{11}a_{23}^2-a_{22}(a_{13}-R_{AC})^2]\right\} \tag{37}$$

where

$$F = [3(m^2-1)+2\lambda^2](\bar{a}_{11}a_{22}-a_{12}^2)+\lambda t^2[a_{22}(a_{13}-R_{AC})-a_{12}a_{23}] + +(\lambda^2/2)\left\{a_{44}(\bar{a}_{11}+a_{22})-t^2[a_{23}^2+(a_{13}-R_{AC})^2]\right\}$$

$$\bar{a}_{11} = \lambda^2+m^2(1+k)(1-\nu)/2$$

and the subscript R.S. means rigid support. For large values of λ equation (37) becomes

$$[\Phi_1]_{R.S.} = \frac{(3-t^2)+k[16\lambda^4+3(m^2-1)^2+8m^2\lambda^2-8\nu\lambda^2]+t^2\lambda^4(1-\nu^2)/(m^2+\lambda^2)^2}{3(m^2-1)+2\lambda^2} \tag{38}$$

For structures having closely spaced rings the value of (a/L) is large, consequently λ is large and expressions (36) and (38) apply. It is also known that the number of half waves n in the axial direction will be small for this type of structure, so that only n = 1, 2 and possibly 3 need be used in the numerical calculations of the buckling load.

The actual procedure is best explained by numerical examples. This is done in the following section.

NUMERICAL EXAMPLES

The ring section and dimensions of the shell used in the numerical example given here are the same as those used in Ref. (1), and are shown in Fig. (1) of the present report. A shell with a radius of 103 inches and thickness of 5/8 inch is considered. The ring spacing is 30 inches and each ring is a 10 inch deep I-section whose flanges are 4-3/4 inch long and 1/2 inch thick and whose web is also 1/2 inch thick. As a second example all dimensions are kept the same except that the flange width is reduced from 4-3/4 inch to 3-3/4 inch.

The constants determined by the dimensions of the shell are

$$\lambda^2 = (n\pi a/L)^2 = (116.341)n^2$$

$$k = h^2/12a^2 = 3.06835 \times 10^{-6}$$

$$K/a = 2.0004 \times 10^5$$

Table (I) lists the values of $[\Phi_1]_{S.S.}$ (equation 36) and $[\Phi_1]_{R.S.}$ (equation 38) for $n = 1$ and 2, and for various values of m. The same quantities are plotted in Figure (2). From Table (I) it is seen that for $n = 1$, the minimum value of $[\Phi_1]_{S.S.}$ is 1.642×10^{-3} while the minimum value of $[\Phi_1]_{R.S.}$ is 2.358×10^{-3} These two values determine a band in Fig. (2) and the minimum for the actual structure with partial constraints must lie within the band.

For $n = 2$, the minimum value of $[\Phi_1]_{S.S.}$ is found to be 4.01×10^{-3} while the minimum value of $[\Phi_1]_{R.S.}$ is 8.607×10^{-3}. These values determine a new band which should contain the minimum for a partially constrained shell with $n = 2$. Since the bands determined by $n = 1$ and $n = 2$ do not overlap, the minimum buckling load parameter Φ_1 will lie in the band determined by $n = 1$. This band also shows that value of m for the actual structure lies between 14 and 18.

These preliminary calculations are valid for both the 3-3/4 and 4-3/4 inch inch flange widths since expressions (36) and (38) do not contain ring frame quantities.

The approximate values of $m = 14$ to 18 in conjunction with $n = 1$ are then used with the more accurate expression (27) which applies to odd values of n to determine the actual buckling load parameter for the structure.

For the ring frame with the 4-3/4 inch flange width, the moment of inertia, warping constant and St. Venant torsion constant are

$$I_{z_o} = 9.025 \text{ in.}^4$$

$$\Gamma = 201.505 \text{ in.}^6$$

$$C_r = .792 \text{ in.}^4$$

The constants defined by equations (13) are

$$M = 8.257 \times 10^{-5}$$
$$N = .279 \times 10^{-5}$$
$$Q = .0174 \times 10^{-5}$$
$$S = .0485$$
$$R_A = m^2(8.275m^2 + .279) \times 10^{-5}$$
$$R_C = \lambda^2 [8.257 + m^2(.0352m^2 - .549)] \times 10^{-5}$$
$$R_{AC} = m^2 \lambda [8.522 - m^2(.384)] \times 10^{-5}$$

In the calculated results listed in Table II, some additional values of m have been included to facilitate the construction of suitable curves. The results for the 4-3/4 inch flange width are given in Fig. ( ). The minimum value of Φ_1 is found to be

$$\Phi_1 = 2.234 \times 10^{-5}$$

The corresponding critical pressure is

$$p_{cr} = 447 \text{ lbs. per sq. in.}$$

For the ring frame with 3-3/4 inch flange width the moment of inertia, warping constant and St. Venant torsion constant are

$$I_{z_0} = 4.488 \text{ in.}^4$$
$$\Gamma = 99.152 \text{ in.}^6$$
$$C_r = .708 \text{ in.}^4$$

The constants defined by equations (13) are

$$M = 4.107 \times 10^{-5}$$
$$N = .249 \times 10^{-5}$$
$$Q = .00855 \times 10^{-5}$$
$$S = .0485$$
$$R_A = m^2(4.115m^2+.249) \times 10^{-5}$$
$$R_C = \lambda^2[4.107+m^2(.0174m^2-.173)] \times 10^{-5}$$
$$R_{AC} = m^2\lambda[4.344-m^2(.191)] \times 10^{-5}$$

Table II lists the values of Φ_1 for different values of m and the results are plotted in Figure (3). The minimum value of Φ_1 is found to be

$$\Phi_1 = 2.150 \times 10^{-3}$$

The corresponding critical pressure is

$$p_{cr} = 430 \text{ lbs. per sq. in.}$$

It is noted that the minimum values $\Phi_1 = 2.234 \times 10^{-3}$ and $\Phi_1 = 2.150 \times 10^{-3}$ obtained above fall within the band $n = 1$ of Fig. (2).

CONCLUSIONS

The critical hydrostatic pressures obtained in Ref. (1) by using the sinusoidal displacement pattern with inflection points at the ring frames were found to be 970 lb. per sq. in. and 756 lb. per sq. in. for the cases of the ring frames with 4-3/4 in. and 3-3/4 in. flange widths respectively. These pressures were large compared to the value of 328 lb. per sq. in. obtained from the von Mises solution (Ref.3) which does not include the effect of the rings.

The corresponding critical pressures obtained in this report by using a displacement pattern which permits the existence of inflection points betweem ring frames are 447 and 430 lb. per sq. in. respectively. The considerable reduction of the critical pressure is due to the more realistic deflection pattern assumed.

It is believed that further changes in the d isplacement pattern will not refine the results to any appreciable extent and that the comparison of the values of 447 and 430 lb. per sq. in. to the 328 lb. per sq. in. of von Mises seems reasonable for the structure assumed.

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2. Love, A.E.H.: A Treatise on the Mathematical Theory of Elasticity, Dover Publications, Fourth Edition, 1944.
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TABLE I

THE VARIATION OF THE BUCKLING LOAD PARAMETER Φ_1 WITH m and n FOR SIMPLY SUPPORTED AND RIGIDLY FIXED ENDS

m	n=1 $[\Phi_1]_{S.S.} \times 10^3$	n=1 $[\Phi_1]_{R.S.} \times 10^3$	n=2 $[\Phi_1]_{S.S.} \times 10^3$	n=2 $[\Phi_1]_{R.S.} \times 10^3$
8	3.949	4.962	5.287	12.857
10	2.588	3.616	4.816	12.105
12	1.937	2.878	4.445	11.371
14	1.681	2.513	4.192	10.699
15	1.6416	2.422	4.108	10.398
16	1.6420	2.372	4.050	10.115
17	1.673	2.358	4.019	9.861
18	1.728	2.372	4.010	9.628
20		2.464	4.054	9.241
22			4.170	8.954
24				8.755
28				8.607
30				8.641
34				8.898

TABLE II

THE VARIATION OF THE BUCKLING LOAD PARAMETER Φ_1 WITH m FOR n = 1 AND PARTIALLY CONSTRAINED ENDS

m	$\Phi_1 \times 10^3$ 4-3/4 inch flange	$\Phi_1 \times 10^3$ 3-3/4 inch flange
10	2.893	2.650
12	2.503	2.320
14	2.298	2.173
16	2.234	2.150
18	2.277	2.215
20	2.376	2.346

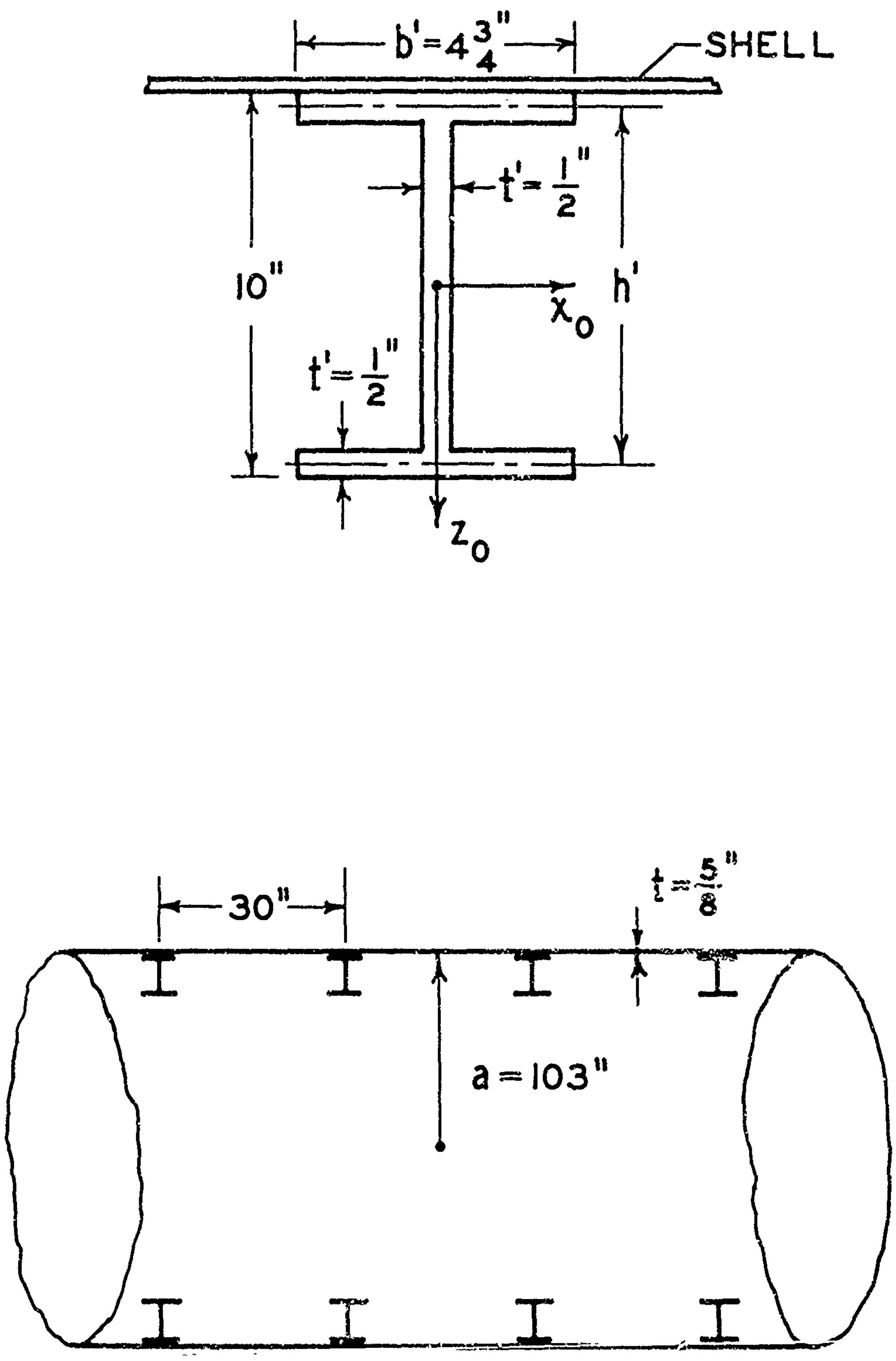

FIG. 1 DIMENSIONS OF RING SECTION AND SHELL

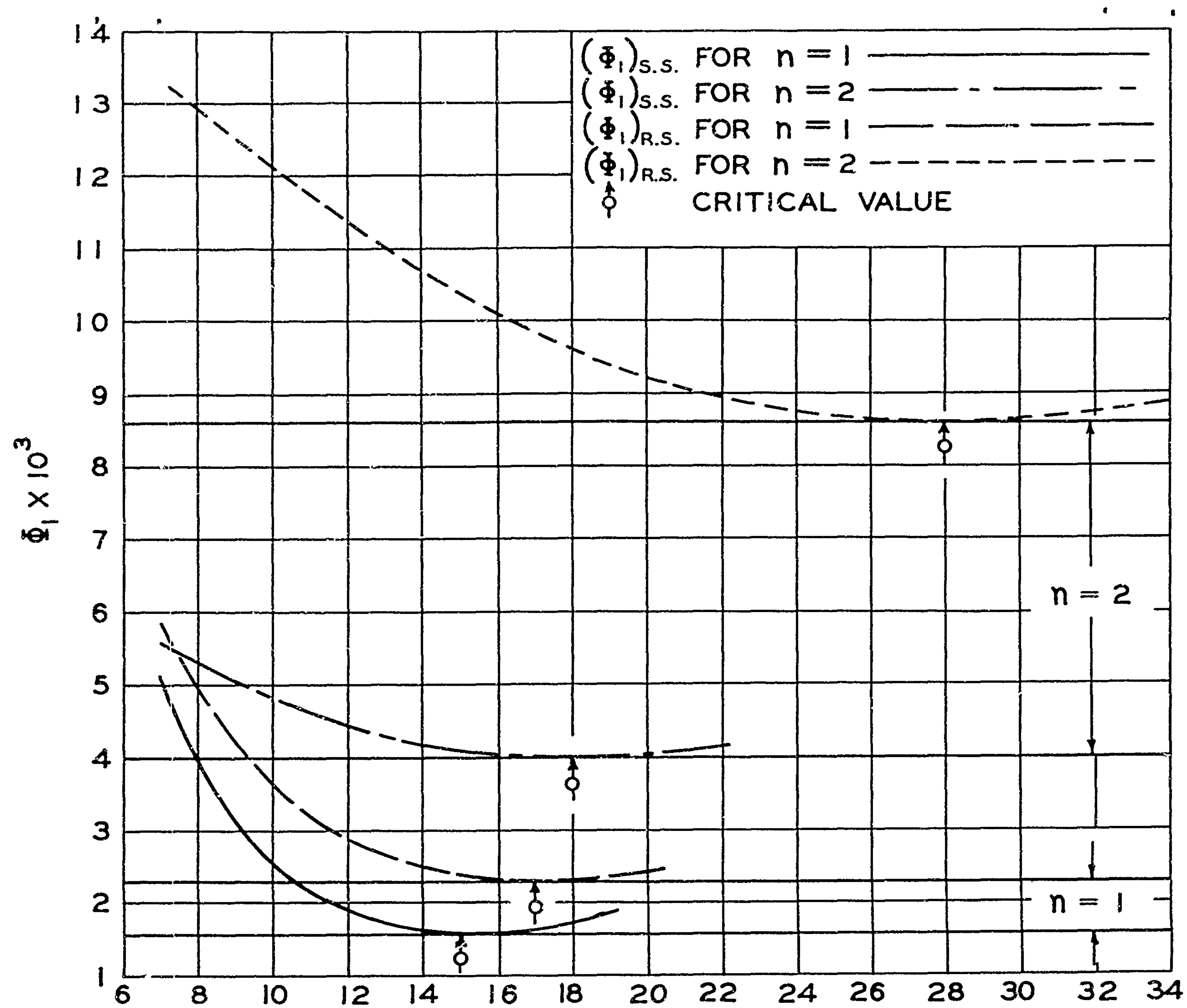

FIG. 2 VARIATION OF BUCKLING LOAD PARAMETER Φ_1 WITH m AND n FOR SIMPLY SUPPORTED AND RIGIDLY FIXED ENDS

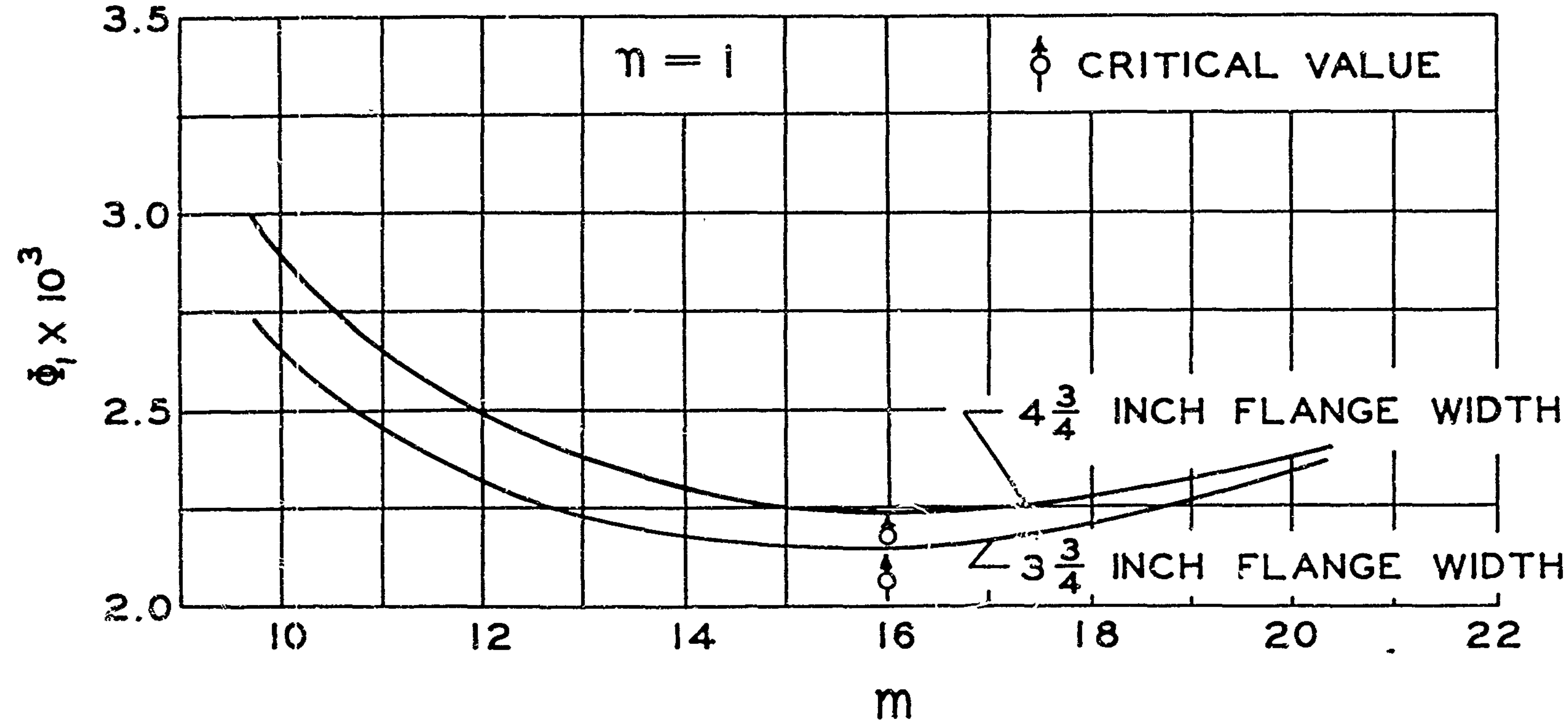

FIG. 3 VARIATION OF BUCKLING LOAD PARAMETER Φ_1 WITH m FOR $n = 1$ FOR SHELL WITH PARTIALLY CONSTRAINED ENDS